

BOUNDARY TRIPLETS FOR SKEW-SYMMETRIC OPERATORS AND THE GENERATION OF STRONGLY CONTINUOUS SEMIGROUPS

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ABSTRACT. We give a self-contained and streamlined exposition of a generation theorem for C_0 -semigroups based on the method of boundary triplets. We apply this theorem to port-Hamiltonian systems where we discuss recent results appearing in stability and control theory. We give detailed proofs and require only a basic knowledge of operator and semigroup theory.

1. INTRODUCTION

It is a prominent result from the theory of unbounded operators that there is a one-to-one correspondence between unbounded self-adjoint and bounded unitary operators. This correspondence is a key ingredient for the characterization of those symmetric operators which allow for a self-adjoint extension, established by von Neumann in 1929.

Given a skew-symmetric operator $A: D(A) \subseteq X \rightarrow X$ on a Hilbert space X , a variant of the correspondence above gives rise to a bijection between dissipative extensions and a certain class of contractive operators. This in turn yields a parametrization of those extensions generating a C_0 -semigroup of contractions

$$\left\{ \begin{array}{l} B: D(B) \rightarrow X \text{ such that } A \subseteq B \text{ \& } B \text{ generates a } C_0\text{-semigroup of contractions} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{l} K: X \rightarrow X \text{ is contractive \&} \\ K(1 - A) = (1 + A) \text{ on } D(A) \end{array} \right\}$$

which was established by Phillips in 1959. Above, the contractions are defined on the initial Hilbert space X . The fact that on $\text{ran}(1 - A)$ the operator K is completely determined by A suggests that a smaller domain can be enough for the parametrization. In addition, classical applications to boundary value problems—where the extension problem amounts to determine the correct boundary conditions yielding a generator—also suggest that the degrees of freedom in the extension problem can be captured by operators defined on a much smaller, possibly even finite-dimensional, space.

Boundary triplets are the technical incarnation of this idea. Beginning in the late 1960's their theory was developed for the case of a symmetric operator A by making use of linear relations in the state-ments as well as in the proofs. We refer to Gorbachuk, Gorbachuk [14, p. 320] for detailed historical information. Nowadays, half a century later, boundary triplets appear in several very active research areas. As a sample of recent papers in this direction we mention Derkach, Malamud [11, 12], Malamud [19], Behrndt, Langer [5, 6], Behrndt, Hassi, de Snoo, Wietsma [4], Arlinskiĭ [1], Derkach, Hassi, Malamud, de Snoo [8, 9, 10] and direct the reader also to the references therein.

Our aim in this article is to give a self-contained and streamlined exposition of the technique of boundary triplets in the case of a skew-symmetric operator A . Our account is mainly based on the material presented in the books of Gorbachuk, Gorbachuk [14, Chapter 3.1] and Schmüdgen [24, Section 14]. The principal difference of our approach is that we develop from the beginning all technical details in the skew-symmetric situation and that we restrict ourselves to dissipative operators instead of dissipative relations. As it turns out, it is then possible to avoid relations also in all proofs. We believe that this approach is of value in particular for beginners in the subject who have a semigroup background or are heading in a semigroup direction.

In addition to a rigorous exposition of boundary triplets in the skew-symmetric situation and without the use of relations, we give a modern application of the method of boundary triplets. Also here

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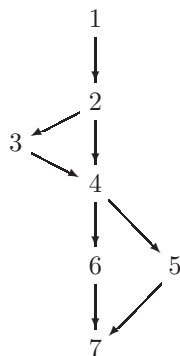
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our exposition is very detailed and intended for beginners in this area. The content is taken from Le Gorrec, Zwart, Maschke [15] who applied the method of boundary triplets to port-Hamiltonian systems. Other applications in systems theory are for instance given in Malinen, Staffans [20], Villegas, Zwart, Le Gorrec, Maschke [26], Arov, Kurula, Staffans [2] or Kurula, Zwart [18]. A general reference for port-Hamiltonian systems is the book by Jacob, Zwart [17] from whence we also took several arguments used in Section 6. Recent results, further improving the current generation theorem, are contained in Augner, Jacob [3] and Jacob, Morris, Zwart [16].

We require the reader to be familiar with the basics of functional analysis as they are usually contained in a first course on this topic, i.e., Banach spaces, linear operators, Hahn-Banach theorem, closed graph and open mapping theorems, Hilbert spaces, orthogonality, Riesz-Fréchet representation theorem etc. For references concerning those basics we use the textbook of Conway [7]. In addition we require some facts about unbounded operators but not more than the definition of the adjoint and some of its consequences. We advise the reader to consult [7, Section X.1, p. 303–308] if necessary. Finally, we require some knowledge on operator semigroups, but again not more than the definition, the notion of a generator and the Hille-Yosida theorem. Here, we recommend to use the introduction given by Rudin [23, p. 375–385].

This article is organized as follows. In Section 2 we first collect basic facts about dissipative operators and the Cayley transform. Then we prove that all dissipative extensions of a given skew-symmetric operator are modulo a sign restrictions of its adjoint. In Section 3 we give a version of the Lumer-Phillips theorem which fits to our situation and whose proof relies only on the prerequisites we mentioned above. Readers familiar with the Lumer-Phillips theorem can skip this section, but are advised to browse the corresponding comments in Section 7. Section 4 contains the parametrization of generators via boundary triplets. The characterization of those operators for which a boundary triplet exists is given in Section 5. Readers who are more focused on applications may also skip this section and move on directly to Section 6 on port-Hamiltonian systems.

The dependencies between the sections are visualized below.



In Sections 2–6 we omit on purpose all references to the original articles. We believe that this makes the text more accessible and easier to read—in particular for beginners. Later, in Section 7, we go through the article again and collect all references omitted before. Moreover, we add comments and remarks on related work.

2. DISSIPATIVE OPERATORS

In what follows we discuss the basic theory of skew-symmetric operators, dissipative extensions and the Cayley transform. We fix a complex Hilbert space X and we assume that its inner product $\langle \cdot, \cdot \rangle$ is linear in the first argument and conjugate linear in the second argument. By a linear operator in X we mean an operator $A: D(A) \rightarrow X$ with a linear subspace $D(A) \subseteq X$ as domain. The symbol “ A ” for an operator comprises that a domain is given and fixed even if not mentioned explicitly. If A is an operator in X and λ a complex number, then we write for simplicity $\lambda - A$ instead of $\lambda \cdot \text{id}_X - A$ and we put $D(\lambda - A) = D(A)$.

Definition 2.1. Let A be a densely defined operator.

- (i) The operator A is *dissipative*, if $\operatorname{Re}\langle Ax, x \rangle \leq 0$ holds for all $x \in D(A)$.
- (ii) The operator A is *maximal dissipative*, if A is dissipative and no proper dissipative extensions of A exists, i.e., if for every operator B with $A \subseteq B$ and B dissipative already $A = B$ follows.

Lemma 2.2. Let A be a densely defined operator.

- (i) If A is dissipative, then A is closable and its closure is again dissipative.
- (ii) If A is maximal dissipative, then A is closed.
- (iii) If A is dissipative, then there exists a maximal dissipative extension B of A .
- (iv) A is dissipative if and only if $\lambda\|x\| \leq \|(\lambda - A)x\|$ holds for every $x \in D(A)$ and every $\lambda > 0$.
- (v) If A is dissipative and closed, then $\operatorname{ran}(\lambda - A) \subseteq X$ is closed for every $\lambda \in \mathbb{R}$.

Proof. (i) Let $(x_n)_{n \in \mathbb{N}} \subseteq D(A)$ and $y \in X$ with $x_n \rightarrow 0$ and $Ax_n \rightarrow y$ be given. For $x \in D(A)$ and $\lambda \in \mathbb{C}$ we have $\operatorname{Re}\langle A(x + \lambda x_n), x + \lambda x_n \rangle \leq 0$ since A is dissipative. Consequently, also for the limit $\operatorname{Re}\langle Ax + \lambda y, x \rangle = \operatorname{Re}\langle Ax, x \rangle + \operatorname{Re}\lambda\langle y, x \rangle \leq 0$ holds. We get that $\operatorname{Re}\lambda\langle y, x \rangle \leq |\operatorname{Re}\langle Ax, x \rangle|$ holds for every $\lambda \in \mathbb{C}$ and every $x \in D(A)$. For fixed y and x the last estimate can only be true for all λ if $\langle y, x \rangle = 0$. But since $D(A) \subseteq X$ is dense the latter in turn can only be true if $y = 0$. This shows that A is closable, see [7, comments after X.1.3]. The closedness together with the continuity of $\operatorname{Re}\langle \cdot, \cdot \rangle$ imply that the closure of A is dissipative.

- (ii) If A is not closed then its closure is a proper dissipative extension in view of (i).
- (iii) The existence of a maximal dissipative extension follows by using Zorn's lemma.
- (iv) “ \Rightarrow ” For $\lambda > 0$ and $x \in D(A)$ we use the dissipativity to get

$$2\lambda\|x\|^2 \leq 2\lambda\langle x, x \rangle - (\langle Ax, x \rangle + \langle x, Ax \rangle) = \langle (\lambda - A)x, x \rangle + \langle x, (\lambda - A)x \rangle \leq 2\|x\|\|(\lambda - A)x\|$$

and obtain $\lambda\|x\| \leq \|(\lambda - A)x\|$ as desired.

“ \Leftarrow ” Let $x \in D(A)$ and $\lambda > 0$ be given. By $\lambda\|x\| \leq \|(\lambda - A)x\|$ we have

$$\|\lambda x\|^2 \leq \|(\lambda - A)x\|^2 = \|\lambda x\|^2 + \|Ax\|^2 - 2\operatorname{Re}\langle Ax, \lambda x \rangle$$

which implies $2\lambda\operatorname{Re}\langle Ax, x \rangle \leq \|Ax\|^2$ and thus $\operatorname{Re}\langle Ax, x \rangle \leq \frac{1}{2\lambda}\|Ax\|^2$. Since $\lambda > 0$ was arbitrary, we proved $\operatorname{Re}\langle Ax, x \rangle \leq 0$.

(v) Fix $\lambda > 0$ and let $(x_n)_{n \in \mathbb{N}} \subseteq D(A)$ with $(\lambda - A)x_n \rightarrow y \in X$ be given. We claim that y is in the range of $\lambda - A$. By (iv) we get $\lambda\|x_n - x_m\| \leq \|(\lambda - A)x_n - (\lambda - A)x_m\|$ for arbitrary $n, m \in \mathbb{N}$ and conclude that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence and thus converges to some $x \in X$. We have $x_n \rightarrow x$ and $(\lambda - A)x_n \rightarrow y$. Since A and therefore also $\lambda - A$ is closed it follows $(\lambda - A)x = y$ and we showed $y \in \operatorname{ran}(\lambda - A)$. \square

Below we consider adjoints of operators. We advise the reader to recall, if necessary, the definition of $A^*: D(A^*) \rightarrow X$ for a given densely defined operator $A: D(A) \rightarrow X$ and to review its basic properties, in particular the fundamental relation $\langle Ax, y \rangle = \langle x, A^*y \rangle$ valid for $x \in D(A)$ and $y \in D(A^*)$, see [7, Section X, §1].

Lemma 2.3. Let A be densely defined, closed and dissipative. Let $\lambda > 0$ and $\varepsilon = \pm 1$ be fixed. Then we have

$$X = \operatorname{ran}(\lambda - \varepsilon A) \oplus \ker(\lambda - \varepsilon A^*).$$

Proof. Straightforward computations based on the definition of the adjoint show that $(\lambda - \varepsilon A)^* = \lambda - \varepsilon A^*$ holds. By [7, Proposition X.1.13] we thus have $\ker(\lambda - \varepsilon A^*) = \operatorname{ran}(\lambda - \varepsilon A)^\perp$. Since $\operatorname{ran}(\lambda - \varepsilon A) \subseteq X$ is closed in view of Lemma 2.2(v) we obtain the desired decomposition by standard Hilbert space theory, see [7, Exercise 3 on p. 11]. \square

In what follows operators of the following type will always be our starting point.

Definition 2.4. A densely defined operator A is said to be *skew-symmetric* if $A \subseteq -A^*$ holds.

Note that skew-symmetric operators are in particular dissipative and thus the properties established for dissipative operators in Lemmas 2.2 and 2.3 are valid for skew-symmetric ones. From now on we stick to the notation $D(-A^*)$ for the domain of $-A^*$, which is by definition equal to $D(A^*)$.

Below, “ \oplus ” denotes a direct, not necessarily orthogonal, sum.

Lemma 2.5. Let A be a densely defined, closed and skew-symmetric operator. Then

$$D(-A^*) = D(A) \oplus \ker(1 - A^*) \oplus \ker(1 + A^*)$$

and $-A^*x = Ax_1 - x_2 + x_3$ holds for $x = x_1 + x_2 + x_3 \in D(-A^*)$ with $x_1 \in D(A)$, $x_2 \in \ker(1 - A^*)$ and $x_3 \in \ker(1 + A^*)$.

Proof. All three summands are subspaces of $D(-A^*)$. We show that their sum is direct. Let $x_1 \in D(A)$, $x_2 \in \ker(1 - A^*)$ and $x_3 \in \ker(1 + A^*)$ satisfy $x_1 + x_2 + x_3 = 0$. We compute $0 = (1 - A^*)(x_1 + x_2 + x_3) = (1 + A)x_1 + (1 + 1)x_2 + 0$. Here, $(1 + A)x_1 \in \text{ran}(1 + A)$ and $2x_2 \in \ker(1 + A^*)$. The latter two spaces are orthogonal. Therefore, $x_2 = 0$ and $(1 + A)x_1 = 0$, i.e.,

$$0 = \langle (1 + A)x_1, x_1 \rangle = \langle x_1, x_1 \rangle + \langle Ax_1, x_1 \rangle.$$

Since A is skew-symmetric we have $\text{Re}\langle Ax_1, x_1 \rangle = 0$. Thus, taking real parts on both sides of the last equation gives $\langle x_1, x_1 \rangle = 0$ and thus $x_1 = 0$. It follows $x_3 = 0$ and we are done.

In order to show $D(-A^*) \subseteq D(A) + \ker(1 - A^*) + \ker(1 + A^*)$ we take $x \in D(-A^*)$ and decompose $(1 + A^*)x \in X$ according to Lemma 2.3 with $\lambda = 1$ and $\varepsilon = 1$, i.e., we take $x_0 \in D(A)$ and $y_0 \in \ker(1 - A^*)$ such that $(1 + A^*)x = (1 - A)x_0 + y_0$. Now we compute

$$\begin{aligned} (1 + A^*)(x - x_0 - \frac{y_0}{2}) &= (1 - A)x_0 + y_0 - (1 + A^*)x_0 - (1 + A^*)\frac{y_0}{2} \\ &= (1 - A)x_0 + y_0 - (1 - A^*)x_0 - (1 + 1)\frac{y_0}{2} \\ &= 0, \end{aligned}$$

which shows $x - x_0 - \frac{y_0}{2} \in \ker(1 + A^*)$. We put $x_1 = -x_0$, $x_2 = -\frac{y_0}{2}$ and $x_3 = x + x_0 + \frac{y_0}{2}$ and thus $x = x_1 + x_2 + x_3$ holds with $x_1 \in D(A)$, $x_2 \in \ker(1 - A^*)$ and $x_3 \in \ker(1 + A^*)$. \square

In [7, Section X, §3] and also in [23, Chapter 13] the reader can find a detailed study of the Cayley transform in the context of symmetric operators. Below we adjust its definition for our case of a skew-symmetric and dissipative operators.

Definition 2.6. Let A be an operator. The *Cayley transform* C_A of A is defined via

$$C_A = (1 + A)(1 - A)^{-1}: \text{ran}(1 - A) \rightarrow X$$

provided that $1 - A: D(A) \rightarrow X$ is injective.

Below we establish properties of C_A .

Lemma 2.7. Let A be densely defined and dissipative. Then the Cayley transform C_A is well-defined, contractive and its range is given by $\text{ran } C_A = \text{ran}(1 + A)$. We have $Ax = -(1 - C_A)(1 + C_A)^{-1}x$ for $x \in D(A)$. If A is skew-symmetric, then C_A is isometric.

Proof. Let $x \in D(A)$ be such that $(1 - A)x = 0$, i.e., $Ax = x$ and thus $0 \geq \text{Re}\langle Ax, x \rangle = \text{Re}\|x\| \geq 0$ which implies $x = 0$. Therefore, $1 - A: D(A) \rightarrow \text{ran}(1 - A)$ is bijective and C_A is well-defined. Let $x \in \text{ran}(1 - A)$, i.e., $x = (1 - A)y$ for some $y \in D(A)$. We estimate

$$\begin{aligned} \|y + Ay\|^2 &= \|y\|^2 + \|Ay\|^2 + \langle Ay, y \rangle + \langle y, Ay \rangle \\ &\leq \|y\|^2 + \|Ay\|^2 - (\langle Ay, y \rangle + \langle y, Ay \rangle) \\ &= \|y - Ay\|^2 \end{aligned}$$

where we used that $\langle Ay, y \rangle + \langle y, Ay \rangle = 2\text{Re}\langle Ay, y \rangle \leq 0$ holds. Since $C_Ax = y + Ay$ and $y - Ay = x$ are valid, we proved $\|C_Ax\| \leq \|x\|$. Finally $C_A(y - Ay) = y + Ay$ holds for every $y \in D(A)$ and all elements of $\text{ran}(1 - A)$ are of the form $y - Ay$ with $y \in D(A)$. Therefore, $\text{ran } C_A = \text{ran}(1 + A)$ holds.

Let $x \in D(A)$ and put $y = (1 - A)x \in \text{ran}(1 - A)$. Then $C_A y = (1 + A)(1 - A)^{-1}(1 - A)x = (1 + A)x$ and therefore

$$(1 + C_A)y = \frac{1}{2}((1 - A)x + (1 + A)x) = x$$

holds. This shows that $1 + C_A: \text{ran}(1 - A) \rightarrow D(A)$ is surjective. Let now $(1 + C_A)y = 0$. By the above it follows $x = 0$ and thus $y = (1 - A)x = 0$. Since every $y \in \text{ran}(1 - A)$ is of the form above, it follows that $1 + C_A: \text{ran}(1 - A) \rightarrow D(A)$ is also injective and consequently $(1 + C_A)^{-1}: D(A) \rightarrow \text{ran}(1 - A)$ exists. Further, with x and y as above, the formula $2(1 + C_A)^{-1}x = y$ is valid. Now we compute

$$\frac{1}{2}(1 - C_A)y = \frac{1}{2}((1 - A)x - (1 + A)x) = -Ax$$

and conclude from this and the above that $Ax = -\frac{1}{2}(1 - C_A)y = -(1 - C_A)(1 + C_A)^{-1}x$ holds as desired for $x \in D(A)$.

If A is skew-symmetric, the first estimate at the beginning of the proof becomes an equality as $\langle Ay, y \rangle + \langle y, Ay \rangle = 0$ holds in this case. Thus, $\|C_A x\| = \|x\|$ follows for every $x \in \text{ran}(1 - A)$. \square

Proposition 2.8. Let A be densely defined, closed and skew-symmetric, i.e., $A \subseteq -A^*$. Let B be a dissipative extension of A . Then $B \subseteq -A^*$ holds.

Proof. By Lemma 2.7 the Cayley transforms

$$C_A = (1 + A)(1 - A)^{-1}: \text{ran}(1 - A) \rightarrow X \quad \text{and} \quad C_B = (1 + B)(1 - B)^{-1}: \text{ran}(1 - B) \rightarrow X$$

are both well-defined, C_B is contractive and C_A is isometric. By construction we have $C_A \subseteq C_B$ and from Lemma 2.3 with $\lambda = 1$ we obtain

$$X = \text{ran}(1 - A) \oplus \ker(1 - A^*) \quad \text{and} \quad X = \text{ran}(1 + A) \oplus \ker(1 + A^*).$$

We claim that there exists $C: D(C) \rightarrow \ker(1 + A^*)$ with $D(C) \subseteq \ker(1 - A^*)$ such that

$$C_B x = C_A x + Cx$$

holds for every $x \in \text{ran}(1 - B)$. For this we have to show $C_B(\text{ran}(1 - B) \cap \ker(1 - A^*)) \subseteq \ker(1 + A^*)$. Let $x \in \text{ran}(1 - B) \cap \ker(1 - A^*)$, i.e., $x \in \text{ran}(1 - B)$ with $x \perp \text{ran}(1 - A)$. Let $y \in \text{ran}(1 - A)$ and $\lambda \in \mathbb{C}$. We have

$$\|\lambda C_B y\|^2 + \|C_B x\|^2 + 2\text{Re}\langle \lambda C_B y, C_B x \rangle = \|C_B(\lambda y + x)\|^2 \leq \|\lambda y + x\|^2 = \|\lambda y\|^2 + \|x\|^2$$

where the estimate holds since C_B is contractive and the last equality holds since $x \perp y$ is valid by our assumptions. Taking $\|C_B y\| = \|C_A y\| = \|y\|$ into account yields

$$2\text{Re}\lambda\langle C_B y, C_B x \rangle \leq \|x\|^2 - \|C_B x\|^2$$

which can only be true for every $\lambda \in \mathbb{C}$, if $\langle C_B y, C_B x \rangle = 0$ holds. We established $C_B x \perp C_B y$ as desired and proved the decomposition.

Now we show that $B \subseteq -A^*$ holds. Let $x \in D(B)$ and put $y = (1 - B)x$. Then we have $C_B y = (1 + B)(1 - B)^{-1}(1 - B)x = (1 + B)x$ and thus

$$\frac{1}{2}(y + C_B y) = \frac{1}{2}((1 - B)x + (1 + B)x) = x.$$

According to the latter two paragraphs we can decompose $y = y_1 + y_2$ with $y_1 \in \text{ran}(1 - A)$ and $y_2 \in \ker(1 - A^*)$ such that $C_B y = C_A y_1 + C y_2$ where $C_A y_1 \in \text{ran}(1 + A)$ and $C y_2 \in \ker(1 + A^*)$ holds. Thus we get

$$\frac{1}{2}(y + C_B y) = \frac{1}{2}(y_1 + y_2 + C_A y_1 + C y_2) = \frac{1}{2}((y_1 + C_A y_1) + y_2 + C y_2).$$

Here, $y_1 + C_A y_1 \in D(A)$: For $y_1 \in \text{ran}(1 - A)$ we select $z \in D(A)$ such that $y_1 = (1 - A)z$ holds and compute $y_1 + C_A y_1 = (1 - A)z + (1 + A)(1 - A)^{-1}(1 - A)z = z \in D(A)$. Combining both computations

we get

$$x = \frac{1}{2}(y_1 + C_A y_1) + \frac{y_2}{2} + \frac{C_A y_2}{2} \in D(A) \oplus \ker(1 - A^*) \oplus \ker(1 + A^*) = D(-A^*)$$

where the last equality follows from Lemma 2.5. This establishes $D(B) \subseteq D(-A^*)$. Using Lemma 2.5 again, we obtain for $x \in D(B)$ as above

$$-A^*x = \frac{1}{2}A(y_1 + C_A y_1) - \frac{y_2}{2} + \frac{C_A y_2}{2} = \frac{y_1}{2} + \frac{C_A y_1}{2} - \frac{y_2}{2} + \frac{C_A y_2}{2}$$

where we used Lemma 2.7 to get $A(y_1 + C_A y_1) = -(1 - C_A)(1 + C_A)^{-1}(1 + C_A)y_1 = C_A y_1 - y_1$. From $C_B y = (1 + B)x = x + Bx$ and $x = \frac{1}{2}(y + C_B y)$ we get on the other hand

$$Bx = C_B y - x = C_B y - \frac{1}{2}(y + C_B y) = \frac{1}{2}(C_B y - y) = \frac{C_A y_1}{2} + \frac{C_A y_2}{2} - \frac{y_1}{2} - \frac{y_2}{2}$$

which shows $Bx = -A^*x$ and finishes the proof. \square

3. THE LUMER-PHILLIPS THEOREM

In this section we give a proof of a version of the Lumer-Phillips theorem which fits precisely to the situation where we apply it in Section 4 and to our terminology of maximal dissipativity. Those readers familiar with one or another version of the Lumer-Phillips theorem are advised to consult the corresponding paragraph in Section 7.

Here, our aim is to give a proof which only relies on the theory of semigroups developed in [23, pp. 375].

Theorem 3.1. Let X be a Hilbert space and let A be densely defined. Then A generates a C_0 -semigroup of contractions on X if and only if A is maximal dissipative.

Proof. “ \Rightarrow ” Let A generate a C_0 -semigroup of contractions. Then by [23, 13.35(f)] the map $\lambda - A$ has an inverse in $L(X)$ for every $\lambda > 0$ and the estimate $\|(\lambda - A)^{-1}y\| \leq \frac{1}{\lambda}\|y\|$ holds for every $\lambda > 0$ and every $y \in X$. By Lemma 2.2(iv) it follows that A is dissipative. In addition it follows that $\lambda - A$ is surjective. Since by Lemma 2.2(iv) every dissipative operator is in particular injective, A cannot have a proper dissipative extension and thus is maximal dissipative.

“ \Leftarrow ” Let A be maximal dissipative. We use the Hille-Yosida theorem [23, 13.37]. Let $\lambda > 0$ be given. By Lemma 2.2(iv) we have $\lambda\|x\| \leq \|(\lambda - A)x\|$ for any $x \in D(A)$. Consequently, $\lambda - A: D(A) \rightarrow X$ is injective. By Lemma 2.2(ii) and Lemma 2.2(v) it follows that $\text{ran}(\lambda - A) \subseteq X$ is closed. Assume that the latter is a proper inclusion. Then we can write $X = \text{ran}(\lambda - A) \oplus \ker(\lambda - A^*)$ with an orthogonal direct sum, cf. Lemma 2.3. We define

$$B: D(B) \rightarrow X \quad \text{with} \quad D(B) = D(A) + \ker(\lambda - A^*) \quad \text{and} \quad B(x_1 + x_2) = Ax_1 - \lambda x_2.$$

The operator B is well-defined: If $x_1 + x_2 = 0$ then $x_2 = -x_1 \in D(A)$ and $Ax_1 = -Ax_2$ together with $A^*x_2 = \lambda x_2$ implies $\langle Ax_1, x_1 \rangle = \langle Ax_2, x_2 \rangle = \langle x_2, A^*x_2 \rangle = \langle x_2, \lambda x_2 \rangle = \lambda \langle x_2, x_2 \rangle \geq 0$. On the other hand $\lambda \langle x_2, x_2 \rangle = \text{Re} \langle Ax_2, x_2 \rangle \leq 0$. Therefore, $x_2 = 0$ and then $x_1 = 0$ follows. For $x_1 \in D(A)$ and $x_2 \in \ker(\lambda - A^*)$ we compute $\langle Ax_1, x_2 \rangle = \langle x_1, A^*x_2 \rangle = \langle x_1, \lambda x_2 \rangle = \lambda \langle x_1, x_2 \rangle$ and obtain

$$\begin{aligned} \langle B(x_1 + x_2), x_1 + x_2 \rangle &= \langle Ax_1, x_1 \rangle + \langle Ax_1, x_2 \rangle + \langle -\lambda x_2, x_1 \rangle + \langle -\lambda x_2, x_2 \rangle \\ &= \langle Ax_1, x_1 \rangle + \lambda \langle x_1, x_2 \rangle - \lambda \langle x_2, x_1 \rangle - \lambda \langle x_2, x_2 \rangle \\ &= \langle Ax_1, x_1 \rangle + i\lambda \text{Im} \langle x_1, x_2 \rangle - \lambda \|x_2\|^2 \end{aligned}$$

which implies $\text{Re} \langle B(x_1 + x_2), x_1 + x_2 \rangle = \text{Re} \langle Ax_1, x_1 \rangle - \lambda \|x_2\|^2 \leq 0$. Thus, B is dissipative. By construction B is a proper extension of A which contradicts the maximal dissipativity of A and thus establishes that $\lambda - A: D(A) \rightarrow X$ is also surjective. But then the estimate which we obtained above from Lemma 2.2(iv) provides $\|(\lambda - A)^{-1}y\| \leq \frac{1}{\lambda}\|y\|$ for every $y \in X$. Finally, for every $n \in \mathbb{N}$ we have $\|(\lambda - A)^{-n}\| \leq \|(\lambda - A)^{-1}\|^n \leq \lambda^{-n}$ which establishes the condition in the Hille-Yosida Theorem [23, 13.37] with $C = 1$ and $\gamma = 0$. The proof of the latter theorem, cf. [23, p. 382, first paragraph] shows that in this case the operators in the semigroup are all contractions. \square

4. BOUNDARY TRIPLETS AND GENERATION

This section is the heart of the current article. Below we give the parametrization of generators via contractions defined on an auxiliary Hilbert space employing a boundary triplet. We mainly follow the account of Gorbachuk, Gorbachuk [14] and Schmüdgen [24] but start with a skew-symmetric operator and completely avoid the use of relations.

Definition 4.1. Let X be a Hilbert space and A be a densely defined and skew-symmetric operator in X . A *boundary triplet* for A is a triple (H, Γ_1, Γ_2) consisting of a Hilbert space H and linear maps $\Gamma_1, \Gamma_2: D(-A^*) \rightarrow H$ such that the two conditions

$$(BT-1) \quad \forall x, y \in D(-A^*): \langle A^*x, y \rangle_X + \langle x, A^*y \rangle_X = \langle \Gamma_1x, \Gamma_2y \rangle_H + \langle \Gamma_2x, \Gamma_1y \rangle_H$$

$$(BT-2) \quad \forall y_1, y_2 \in H \exists x \in D(-A^*): \Gamma_1x = y_1 \text{ and } \Gamma_2x = y_2$$

are satisfied.

We note that in the literature the above triple—or its symmetric counterpart—is usually referred to as a boundary triplet “for A^* ”. The condition (BT-1) is often said to be an “abstract Green identity”. Now we state the main result of this section.

Theorem 4.2. Let A be a densely defined, closed, skew-symmetric operator in a Hilbert space X and let (H, Γ_1, Γ_2) be a boundary triplet for A . Let B be an extension of A . Then B generates a C_0 -semigroup of contractions on X if and only if there exists a uniquely determined $K \in L(H)$ with $\|K\|_{L(H)} \leq 1$ such that $D(B) = \{x \in X; (K - 1)\Gamma_1x - (K + 1)\Gamma_2x = 0\}$ and $B = -A^*|_{D(B)}$.

Note that we below *explicitly* construct a bijection and its inverse. For applications the technical version, Proposition 4.8, of the theorem above can be more handy.

For the rest of this section we fix a densely defined, closed and skew-symmetric operator A and a boundary triplet (H, Γ_1, Γ_2) for A . We start with the following fundamental relations between operators and boundary triplets.

Lemma 4.3. Let A be as above.

- (i) $D(A) = \{x \in D(-A^*); \Gamma_1x = \Gamma_2x = 0\}$.
- (ii) $\forall x \in D(-A^*): \operatorname{Re}\langle -A^*x, x \rangle_X = -\operatorname{Re}\langle \Gamma_1x, \Gamma_2x \rangle_H$.

Proof. (i) “ \subseteq ” Let $x \in D(A)$. We consider the points Γ_2x and Γ_1x in H and select $y \in D(A^*)$ with $\Gamma_1y = \Gamma_2x$ and $\Gamma_2y = \Gamma_1x$. According to (BT-1) we have

$$\langle A^*x, y \rangle_X + \langle x, A^*y \rangle_X = \langle \Gamma_1x, \Gamma_1x \rangle_H + \langle \Gamma_2x, \Gamma_2x \rangle_H = \|\Gamma_1x\|_H^2 + \|\Gamma_2x\|_H^2$$

where the left-hand side is zero since $A^*x = -Ax$. Thus, $\Gamma_1x = \Gamma_2x = 0$ follows.

“ \supseteq ” Let $x \in D(-A^*)$ be given with $\Gamma_1x = \Gamma_2x = 0$. Then the right-hand side of (BT-1) is zero for any $y \in D(-A^*)$ and we obtain by conjugation

$$\langle A^*y, x \rangle_X = -\langle y, A^*x \rangle_X$$

for all $y \in D(-A^*)$. Thus, the map $D(-A^*) \rightarrow \mathbb{C}$, $y \mapsto \langle A^*y, x \rangle_X$ is continuous and consequently $x \in D((-A^*)^*)$. As A is closed, [7, X.1.6] yields $A^{**} = A$ and thus $x \in D((-A^*)^*) = D(A^{**}) = D(A)$ follows.

(ii) Let $x \in D(-A^*)$. We use (BT-1) to compute

$$2\operatorname{Re}\langle A^*x, x \rangle_X = \langle A^*x, x \rangle_X + \langle x, A^*x \rangle_X = \langle \Gamma_1x, \Gamma_2x \rangle_H + \langle \Gamma_2x, \Gamma_1x \rangle_H = 2\operatorname{Re}\langle \Gamma_1x, \Gamma_2x \rangle_H$$

which shows the equation in (ii). \square

The next lemma contains a first characterization of dissipative extensions via an associated boundary triplet in terms of an estimate.

Lemma 4.4. Let B be an operator with $A \subseteq B \subseteq -A^*$. Then B is dissipative if and only if $\|\Gamma_1x + \Gamma_2x\|_H \leq \|\Gamma_1x - \Gamma_2x\|_H$ holds for all $x \in D(B)$.

Proof. For $x \in D(-A^*)$ and $\varepsilon = \pm 1$ we compute

$$\begin{aligned}\|\Gamma_1 x + \varepsilon \Gamma_2 x\|_H^2 &= \langle \Gamma_1 x + \varepsilon \Gamma_2 x, \Gamma_1 x + \varepsilon \Gamma_2 x \rangle_H \\ &= \langle \Gamma_1 x, \Gamma_1 x \rangle_H + \langle \Gamma_1 x, \varepsilon \Gamma_2 x \rangle_H + \langle \varepsilon \Gamma_2 x, \Gamma_1 x \rangle_H + \langle \varepsilon \Gamma_2 x, \varepsilon \Gamma_2 x \rangle_H \\ &= \|\Gamma_1 x\|_H^2 + \|\Gamma_2 x\|_H^2 + \varepsilon 2 \operatorname{Re} \langle \Gamma_1 x, \Gamma_2 x \rangle_H.\end{aligned}$$

Subtracting the two equations for $\varepsilon = \pm 1$, using Lemma 4.3(ii) and $B \subseteq -A^*$ we obtain that

$$\|\Gamma_1 x + \Gamma_2 x\|_H^2 - \|\Gamma_1 x - \Gamma_2 x\|_H^2 = -4 \operatorname{Re} \langle \Gamma_1 x, \Gamma_2 x \rangle_H = 4 \operatorname{Re} \langle Bx, x \rangle_X$$

holds for every $x \in D(B)$. B is dissipative if and only if the above is less or equal zero for every $x \in D(B)$. And this is equivalent to $\|\Gamma_1 x + \Gamma_2 x\|_H \leq \|\Gamma_1 x - \Gamma_2 x\|_H$ for every $x \in D(B)$. \square

Our aim is now to parametrize the estimate in Lemma 4.4 by a contractive operator. For a dissipative extension B of A we therefore define

$$\begin{aligned}\phi(B)(\Gamma_1 x - \Gamma_2 x) &= \Gamma_1 x + \Gamma_2 x \\ D(\phi(B)) &= \{\Gamma_1 x - \Gamma_2 x; x \in D(B)\}\end{aligned}\tag{1}$$

which is a contractive operator in H by the following lemma. Here we call $K: D(K) \rightarrow H$ *contractive* if $\|Kx\|_H \leq \|x\|_H$ holds for all $x \in D(K)$. The term *contraction* is reserved for those contractive K which are defined on the whole space.

Lemma 4.5. The operator $\phi(B): D(\phi(B)) \rightarrow H$ is well-defined and contractive.

Proof. Let $x, y \in D(B)$ with $\Gamma_1 x - \Gamma_2 x = \Gamma_1 y - \Gamma_2 y$ be given. Put $z = x - y$. Then $\Gamma_1 z = -\Gamma_2 z$ holds. Since Proposition 2.8 implies $B \subseteq -A^*$ we can use Lemma 4.3(ii) and $B \subseteq -A^*$ to compute

$$\operatorname{Re} \langle Bz, z \rangle_X = -\operatorname{Re} \langle \Gamma_1 z, \Gamma_2 z \rangle_H = \operatorname{Re} \langle \Gamma_1 z, \Gamma_1 z \rangle_H = \|\Gamma_1 z\|_H^2.$$

The latter is greater or equal to zero in view of the norm and less or equal to zero since B is dissipative. Thus, $-\Gamma_2 z = \Gamma_1 z = 0$ and therefore $\Gamma_1 x = \Gamma_1 y$ and $\Gamma_2 x = \Gamma_2 y$. This shows that $\phi(B)$ is well defined. It is then clear that $\phi(B)$ is linear.

To show the estimate, let $y \in D(\phi(B))$ be given. By definition there is $x \in D(B)$ such that $y = \Gamma_1 x - \Gamma_2 x$ holds. The dissipativity of B yields by Lemma 4.4 the estimate

$$\|\phi(B)y\|_H = \|\phi(B)(\Gamma_1 x - \Gamma_2 x)\|_H = \|\Gamma_1 x + \Gamma_2 x\|_H \leq \|\Gamma_1 x - \Gamma_2 x\|_H = \|y\|_H$$

which shows that $\phi(B)$ is contractive. \square

The above yields a map ϕ which assigns to a given dissipative extension a contractive operator. Next we consider a contractive operator $K: D(K) \rightarrow H$ and define

$$\begin{aligned}\psi(K) &= -A^*|_{D(\psi(K))} \\ D(\psi(K)) &= \{x \in D(-A^*); \Gamma_1 x - \Gamma_2 x \in D(K) \text{ and } K(\Gamma_1 x - \Gamma_2 x) = \Gamma_1 x + \Gamma_2 x\}\end{aligned}\tag{2}$$

which is a dissipative operator in X that extends A by the following lemma.

Lemma 4.6. The operator $\psi(K): D(\psi(K)) \rightarrow X$ is a dissipative extension of A .

Proof. We have $\psi(K) \subseteq -A^*$ by definition. Let $x \in D(A)$, i.e., $\Gamma_1 x = \Gamma_2 x = 0$ by Lemma 4.3(i) and thus $x \in D(\psi(K))$ holds. Therefore we have $A \subseteq \psi(K) \subseteq -A^*$.

Let $x \in D(\psi(K))$. Then $K(\Gamma_1 x - \Gamma_2 x) = \Gamma_1 x + \Gamma_2 x$ holds. Since K is contractive it follows in particular $\|\Gamma_1 x + \Gamma_2 x\|_H \leq \|\Gamma_1 x - \Gamma_2 x\|_H$. Thus $\psi(K)$ is dissipative by Lemma 4.4(ii). \square

We showed that the map ψ assigns to a given contractive operator K a dissipative extension. Below we investigate the relation between the maps ϕ and ψ .

Proposition 4.7. (i) We have $\psi \circ \phi(B) = B$ for every dissipative extension B of A .

(ii) We have $\phi \circ \psi(K) = K$ for every contractive K in H .

Proof. (i) Let B be a dissipative extension of A . Then $\psi \circ \phi(B) = -A^*|_{D(\psi \circ \phi(B))}$ and since $B \subseteq -A^*$ by Proposition 2.8 it is enough to show that $D(\psi \circ \phi(B)) = D(B)$ holds. We have

$$D(\psi \circ \phi(B)) = \{x \in D(-A^*) ; \Gamma_1 x - \Gamma_2 x \in D(\phi(B)) \text{ and } \phi(B)(\Gamma_1 x - \Gamma_2 x) = \Gamma_1 x + \Gamma_2 x\}.$$

In view of (1) the second condition holds provided that $x \in D(B)$. The first condition holds if and only if $x \in D(B)$. Therefore the above set equals $D(B)$ and we are done.

(ii) Let $K: D(K) \rightarrow H$ be contractive. We have

$$\begin{aligned} D(\phi \circ \psi(K)) &= \{\Gamma_1 x - \Gamma_2 x ; x \in D(\psi(K))\} \\ &= \{\Gamma_1 x - \Gamma_2 x ; x \in D(-A^*), \Gamma_1 x - \Gamma_2 x \in D(K) \text{ and } K(\Gamma_1 x - \Gamma_2 x) = \Gamma_1 x + \Gamma_2 x\} \end{aligned}$$

and clearly $D(\phi \circ \psi(K)) \subseteq D(K)$ holds. To show the equality let $y \in D(K)$ be given. By (BT-2) we find $x \in D(-A^*)$ such that

$$\Gamma_1 x = \frac{1}{2}(y + Ky) \text{ and } \Gamma_2 x = \frac{1}{2}(Ky - y).$$

We compute $\Gamma_1 x - \Gamma_2 x = y \in D(K)$ and $\Gamma_1 x + \Gamma_2 x = Ky$. Thus, $K(\Gamma_1 x - \Gamma_2 x) = \Gamma_1 x + \Gamma_2 x$ and we have $y \in D(\phi \circ \psi(K))$. Let now $\Gamma_1 x - \Gamma_2 x$ be an element of $D(\phi \circ \psi(K))$. Then we have $K(\Gamma_1 x - \Gamma_2 x) = \Gamma_1 x + \Gamma_2 x = (\phi \circ \psi(K))(\Gamma_1 x - \Gamma_2 x)$ which shows $K = \phi \circ \psi(K)$. \square

By Proposition 4.7 we get a one-to-one correspondence between the dissipative extensions of A on X and the contractive operators on H

$$\left\{ \begin{array}{l} B: D(B) \subseteq X \rightarrow X \\ A \subseteq B \text{ \& } B \text{ dissipative} \end{array} \right\} \xrightleftharpoons[\psi]{\phi} \left\{ \begin{array}{l} K: D(K) \subseteq H \rightarrow H \\ K \text{ contractive} \end{array} \right\} \quad (3)$$

where the maps ϕ and ψ are defined as above.

Theorem 4.2 states that if K is everywhere defined, then we recover the whole of K and that the operators we get in X are then precisely those which are maximal dissipative. Using the maps ϕ and ψ on the new domains

$$\begin{aligned} \mathcal{D} &= \{B: D(B) \rightarrow X ; B \text{ is a maximal dissipative extension of } A\} \\ \mathcal{K} &= \{K \in L(H) ; K \text{ is a contraction on } H\} \end{aligned} \quad (4)$$

the following proposition together with the Lumer-Phillips Theorem, see Section 3, implies immediately the statement of Theorem 4.2.

Proposition 4.8. The map $\phi: \mathcal{D} \rightarrow \mathcal{K}$ is bijective with inverse $\psi: \mathcal{K} \rightarrow \mathcal{D}$.

Proof. The proof is divided into several steps.

(i) If $K \in \mathcal{K}$ and $\psi(K) \subseteq B$ is dissipative, then we have $\phi(B) = \phi(\psi(K))$.

Indeed, we observe

$$D(\phi(\psi(K))) = \{\Gamma_1 x - \Gamma_2 x ; x \in D(\psi(K))\} \text{ and } \phi(\psi(K))(\Gamma_1 x - \Gamma_2 x) = \Gamma_1 x + \Gamma_2 x$$

and by Proposition 4.7 we have $\phi(\psi(K)) = K$. Since $\psi(K) \subseteq B$ holds by assumption, it follows $\phi(\psi(K)) \subseteq \phi(B)$. As $D(\phi(\psi(K))) = D(K) = H$ is the whole space, we obtain $\phi(B) = \phi(\psi(K))$.

(ii) The map $\psi: \mathcal{K} \rightarrow \mathcal{D}$ is well-defined.

Let $K \in \mathcal{K}$ be given. Lemma 4.6 yields that $\psi(K)$ is dissipative; we have to show that it is maximal dissipative. Assume that B is a dissipative extension of $\psi(K)$, i.e., $\psi(K) \subseteq B$. We claim $\psi(K) = B$. Let $x \in D(B)$. That is, $\Gamma_1 x - \Gamma_2 x \in D(\phi(B))$. By (i) and Proposition 4.7 we have $\phi(B) = K$ and it follows $\Gamma_1 x - \Gamma_2 x \in D(K)$. Moreover, (i) implies $K = \phi(\psi(K))$ and thus

$$\Gamma_1 x + \Gamma_2 x = \phi(\psi(K))(\Gamma_1 x - \Gamma_2 x) = K(\Gamma_1 x - \Gamma_2 x).$$

In view of (2) it follows $x \in D(\psi(K))$.

(iii) If $B \in \mathcal{D}$ and $\phi(B) \subseteq K$ with $K \in \mathcal{K}$, then we have $\phi(B) = \phi(\psi(K)) = K$.

We claim $B \subseteq \psi(K)$. Let $x \in D(B)$. That is, $\Gamma_1 x - \Gamma_2 x \in D(\phi(B))$. Since $\phi(B) \subseteq K$ holds by assumption and by (1) we get

$$\Gamma_1 x + \Gamma_2 x = \phi(B)(\Gamma_1 x - \Gamma_2 x) = K(\Gamma_1 x - \Gamma_2 x).$$

Thus, $x \in D(\psi(K))$ and $\psi(K)x = -A^*x = Bx$ holds by Proposition 2.8. Since B is maximal dissipative and $\psi(K)$ is dissipative by Lemma 4.6 it follows $B = \psi(K)$. But now the conclusion follows from (i).

(iv) The map $\phi: \mathcal{D} \rightarrow \mathcal{K}$ is well-defined.

Let $B \in \mathcal{D}$ be given. We show by contradiction that $D(\phi(B)) = H$ holds. Assume that this is not the case. By Lemma 4.5 and the Hahn-Banach Theorem we can extend $\phi(B)$ to an operator $K \in L(H)$ with $\|K\|_{L(H)} \leq 1$, i.e., $K \in \mathcal{K}$. By (iii) it follows $\phi(B) = K$ but by our assumptions $\phi(B) \subset K$ is a proper subset. The contradiction shows that $\phi(B) \in \mathcal{K}$ holds.

We established that $\phi: \mathcal{D} \rightarrow \mathcal{K}$ and $\psi: \mathcal{K} \rightarrow \mathcal{D}$ are both well-defined. Thus it follows from 4.7 that $\psi \circ \phi = \text{id}_{\mathcal{D}}$ and $\phi \circ \psi = \text{id}_{\mathcal{K}}$ are valid, which finishes the proof. \square

5. EXISTENCE OF BOUNDARY TRIPLETS

In this section we characterize those densely defined, closed and skew-symmetric operators A for which a boundary triplet exists. We note that the applications in Section 6 are independent of the following. The reader can thus proceed directly with the port-Hamiltonian systems, if he or she likes to.

Theorem 5.1. Let A be a densely defined, closed and skew-symmetric operator. Then there exists a boundary triplet (H, Γ_1, Γ_2) for A if and only if $\dim \ker(1 - A^*) = \dim \ker(1 + A^*)$ holds.

Note that the spaces $\ker(1 - A^*)$ and $\ker(1 + A^*)$ are closed, cf. the proof of Lemma 2.3. In the situation of symmetric operators the dimensions of the spaces corresponding to $\ker(1 - A^*)$ and $\ker(1 + A^*)$ are usually called the *deficiency indices* of A . Below, we first prove that their equality is sufficient for the existence of a boundary triplet. This follows immediately from the decomposition established in Lemma 2.5 by computation.

Proposition 5.2. Let A be a densely defined, closed and skew-symmetric operator. If $\dim \ker(1 - A^*) = \dim \ker(1 + A^*)$ holds, then there exists a boundary triplet (H, Γ_1, Γ_2) for A .

Proof. By Lemma 2.5 we have $D(-A^*) = D(A) \oplus \ker(1 - A^*) \oplus \ker(1 + A^*)$ and we denote by

$$P_1: D(-A^*) \rightarrow \ker(1 - A^*) \quad \text{and} \quad P_2: D(-A^*) \rightarrow \ker(1 + A^*)$$

the corresponding projections. Since $\dim \ker(1 - A^*) = \dim \ker(1 + A^*)$, we find an isometric isomorphism $\varphi: \ker(1 + A^*) \rightarrow \ker(1 - A^*)$. We put $H = \ker(1 - A^*)$ endowed with the inner product induced by X , define

$$\Gamma_1 = P_1 + \varphi P_2 \quad \text{and} \quad \Gamma_2 = P_1 - \varphi P_2$$

and check the conditions in Definition 4.1. Let $x, y \in D(-A^*)$ be given. According to the above we have $x = x_0 + P_1 x + P_2 x$ and $y = y_0 + P_1 y + P_2 y$ with $x_0, y_0 \in D(A)$ and the equalities $A^* P_1 x = P_1 x$ and $A^* P_2 x = -P_2 x$. Using the latter, the fact that A is skew-symmetric, and that φ is isometric, a long but straightforward computation shows

$$\begin{aligned} \langle A^* x, y \rangle_X + \langle x, A^* y \rangle_X &= \langle P_1 x, P_1 y \rangle_X - \langle P_2 x, P_2 y \rangle_X + \langle P_1 x, P_1 y \rangle_X - \langle P_2 x, P_2 y \rangle_X \\ &= \langle P_1 x + \varphi P_2 x, P_1 y - \varphi P_2 y \rangle_H + \langle P_1 x - \varphi P_2 x, P_1 y + \varphi P_2 y \rangle_H \\ &= \langle \Gamma_1 x, \Gamma_2 y \rangle_H + \langle \Gamma_2 x, \Gamma_1 y \rangle_H \end{aligned}$$

which proves (BT-1). Let $y_1, y_2 \in H$ be given. We put $x = x_0 + x_1 + x_2 \in D(-A^*)$ with $x_0 = 0 \in D(A)$, $x_1 = \frac{1}{2}(y_1 + y_2) \in H = \ker(1 - A^*)$ and $x_2 = \frac{1}{2}\varphi^{-1}(y_1 - y_2) \in \ker(1 + A^*)$. We get $\Gamma_1 x = x_1 + \varphi x_2 =$

$\frac{1}{2}(y_1 + y_2) + \frac{1}{2}(y_1 - y_2) = y_1$ and similarly $\Gamma_2 x = x_1 - \varphi x_2 = \frac{1}{2}(y_1 + y_2) - \frac{1}{2}(y_1 - y_2) = y_2$ which finishes the proof of (BT-2). \square

The necessity part of Theorem 5.1 uses the main result of Section 4 in a special case and needs the following preparation.

Lemma 5.3. Let A be a densely defined, closed and skew-symmetric operator and let (H, Γ_1, Γ_2) be a boundary triplet for A . We consider $K = -\text{id}_H$. Then the operator $B = \psi(K)$ satisfies $D(B) = \ker \Gamma_1$ and $B = -B^*$.

Proof. We first show that $D(B) = \ker \Gamma_1$ is true.

“ \subseteq ” Let $x \in D(B)$. That is

$$\Gamma_1 x + \Gamma_2 x = K(\Gamma_1 x - \Gamma_2 x) = K(\Gamma_1 x) - K(\Gamma_2 x) = -\Gamma_1 x + \Gamma_2 x.$$

Consequently, $\Gamma_1 x = -\Gamma_1 x$ which implies $x \in \ker \Gamma_1$.

“ \supseteq ” Let $x \in \ker \Gamma_1$, i.e., $\Gamma_1 x = 0$ and thus $K(\Gamma_1 x - \Gamma_2 x) = -K(\Gamma_2 x) = 0 - (-\Gamma_2 x) = \Gamma_1 x + \Gamma_2 x$, which means $x \in D(B)$.

Now we show $B = -B^*$. By Proposition 4.7 we know that B is a maximal dissipative extension of A and thus Proposition 4.7 gives $A \subseteq B \subseteq -A^*$. Therefore, $(-A^*)^* \subseteq B^* \subseteq A^*$ and since $(-A^*)^* = A$, due to the closedness of A , and [7, X.1.6], thus finally $A \subseteq -B^* \subseteq -A^*$ holds.

It remains to show that $D(-B^*) = D(B)$ holds. We know already that both sets are contained in $D(-A^*)$ and thus we fix $y \in D(-A^*)$. In view of $B^* \subseteq A^*$ we get that $y \in D(-B^*)$ holds if and only if $\langle Bx, y \rangle_X = \langle x, A^* y \rangle_X$ is true for every $x \in D(B)$. Using first $B \subseteq -A^*$ and then (BT-1) we can replace the equality in the statement with $\langle -A^* x, y \rangle_X = \langle x, A^* y \rangle_X$. With (BT-1) the latter means exactly $\langle \Gamma_1 x, \Gamma_2 y \rangle_H + \langle \Gamma_2 x, \Gamma_1 y \rangle_H = 0$. We thus have

$$\begin{aligned} D(-B^*) &= \{y \in D(-A^*) : \forall x \in D(B) : \langle \Gamma_1 x, \Gamma_2 y \rangle_H + \langle \Gamma_2 x, \Gamma_1 y \rangle_H = 0\} \\ &= \{y \in D(-A^*) : \forall x \in D(B) : \langle \Gamma_1 x - \Gamma_2 x, \Gamma_1 y \rangle_H = 0\} \\ &= \{y \in D(-A^*) : \forall h \in H : \langle h, \Gamma_1 y \rangle_H = 0\} \\ &= \ker \Gamma_1, \end{aligned}$$

where we used $D(B) = \ker \Gamma_1$ and the fact that $H = D(\psi(B)) = D(K)$, which follows from Proposition 4.7. Thus we have $D(B) = \ker \Gamma_1 = D(-B^*)$. \square

We note that in the literature an operator B with $B = -B^*$ is said to be *skew-adjoint*.

Lemma 5.4. Let A be a densely defined, closed and skew-symmetric operator. Let $B : D(B) \rightarrow X$ be a maximal dissipative extension. Assume that $B = -B^*$ holds. Then $D(-A^*) = D(B) \oplus \ker(1 + \varepsilon A^*)$ holds for $\varepsilon = \pm 1$.

Proof. Let $\varepsilon = \pm 1$. We have $\ker(1 + \varepsilon A^*) \subseteq D(A^*)$ and $D(B) \subseteq D(A^*)$. We observe that $\text{Re} \langle Bx, x \rangle_X = 0$ holds for every $x \in D(B)$ since $B = -B^*$.

Let $x \in D(B) \cap \ker(1 + \varepsilon A^*)$, i.e., $A^* x = -\varepsilon x$ and $Bx = -A^* x = \varepsilon x$. We get

$$0 = \text{Re} \langle Bx, x \rangle_X = \varepsilon \text{Re} \langle x, x \rangle_X = \varepsilon \|x\|_X^2$$

and therefore $x = 0$. Thus we showed that the sum $D(B) + \ker(1 + \varepsilon A^*)$ is direct.

It remains to prove $D(-A^*) = D(B) + \ker(1 + \varepsilon A^*)$.

“ \supseteq ” Since B is dissipative we have $B \subseteq -A^*$ by Proposition 2.8.

“ \subseteq ” We first show that $1 - \varepsilon B : D(B) \rightarrow X$ is bijective. Let $x \in D(B)$ be such that $(1 - \varepsilon B)x = 0$, i.e., $Bx = \varepsilon x$. Then $0 = \text{Re} \langle Bx, x \rangle_X = \varepsilon \|x\|_X^2$ implies $x = 0$. Thus, $\ker(1 - \varepsilon B) = 0$ and $1 - \varepsilon B$ is injective. From Lemma 2.3 we get $X = \text{ran}(1 - \varepsilon B) \oplus \ker(1 - \varepsilon B^*)$ where $\ker(1 - \varepsilon B^*) = \ker(1 + \varepsilon B) = 0$. Thus, $1 - \varepsilon B : D(B) \rightarrow X$ is also surjective.

Now we take $x \in D(-A^*)$ and decompose $(1 + \varepsilon A^*)x$ according to

$$X = \text{ran}(1 - \varepsilon A) \oplus \ker(1 - \varepsilon A^*),$$

cf. Lemma 2.3. Accordingly, we find $x_0 \in D(A)$ and $y_1 \in \ker(1 - \varepsilon A^*)$ such that $(1 + \varepsilon A^*)x = (1 - \varepsilon A)x_0 + y_1$. We put $y_2 = x - x_0 - (1 - \varepsilon B)^{-1}y_1$ and compute

$$\begin{aligned} (1 + \varepsilon A^*)y_2 &= (1 + \varepsilon A^*)x - (1 + \varepsilon A^*)x_0 - (1 + \varepsilon A^*)(1 - \varepsilon B)^{-1}y_1 \\ &= (1 + \varepsilon A^*)x - (1 - \varepsilon A)x_0 - (1 - \varepsilon B)(1 - \varepsilon B)^{-1}y_1 \\ &= y_1 - y_1 = 0 \end{aligned}$$

that is $y_2 \in \ker(1 + \varepsilon A^*)$. On the other hand, $x_0 \in D(A) \subseteq D(B)$ and $(1 - \varepsilon B)^{-1}y_1 \in D(B)$ imply $x_0 + (1 - \varepsilon B)^{-1}y_1 \in D(B)$ and thus

$$x = x_0 + (1 - \varepsilon B)^{-1}y_1 + y_2 \in D(B) + \ker(1 + \varepsilon A^*)$$

as desired. \square

The last ingredient for the proof of the necessity part is the following.

Lemma 5.5. Let A be a densely defined, closed and skew-symmetric operator and let (H, Γ_1, Γ_2) be a boundary triplet for A . Then the map $\Gamma_1: D(-A^*) \rightarrow H$ is continuous when $D(-A^*)$ is endowed with the graph norm $\|\cdot\|_{D(-A^*)}$ defined by $\|x\|_{D(-A^*)} := \|x\|_X + \|-A^*x\|_X$ for $x \in D(-A^*)$.

Proof. We show that the map

$$\Gamma: (D(-A^*), \|\cdot\|_{D(-A^*)}) \rightarrow H \times H, x \mapsto (\Gamma_1 x, \Gamma_2 x)$$

is continuous where we use the product topology on $H \times H$. We apply the version [7, III.12.7] of the closed graph theorem to show the latter. Let $(x_n)_{n \in \mathbb{N}} \subseteq D(-A^*)$ be given with $x_n \rightarrow 0 \in D(-A^*)$ with respect to $\|\cdot\|_{D(-A^*)}$ and $\Gamma_1 x_n \rightarrow y_1, \Gamma_2 x_n \rightarrow y_2$ for $n \rightarrow \infty$ with $y_1, y_2 \in H$. Then, for arbitrary $y \in D(-A^*)$, we can compute

$$\begin{aligned} \langle y_1, \Gamma_2 y \rangle_H + \langle y_2, \Gamma_1 y \rangle_H &= \lim_{n \rightarrow \infty} [\langle \Gamma_1 x_n, \Gamma_2 y \rangle_H + \langle \Gamma_2 x_n, \Gamma_1 y \rangle_H] \\ &= \lim_{n \rightarrow \infty} [\langle A^* x_n, y \rangle_X + \langle x_n, A^* y \rangle_X] \\ &= \langle \lim_{n \rightarrow \infty} A^* x_n, y \rangle_X + \langle \lim_{n \rightarrow \infty} x_n, A^* y \rangle_X \\ &= 0 \end{aligned}$$

since $A^*: (D(-A^*), \|\cdot\|_{D(-A^*)}) \rightarrow X$ is continuous. According to (BT-2) we select $y \in D(-A^*)$ such that $\Gamma_1 y = 0$ and $\Gamma_2 y = y_1$. Then we get from the above

$$0 = \langle y_1, \Gamma_2 y \rangle_H + \langle y_2, \Gamma_1 y \rangle_H = \|y_1\|_X^2$$

and thus $y_1 = 0$. Analogously, it follows $y_2 = 0$ if we select $y \in D(-A^*)$ such that $\Gamma_1 y = y_2, \Gamma_2 y = 0$ and use the formula above. Thus, Γ is continuous and whence also Γ_1 as this map equals the composition of Γ and the projection on the first component of $H \times H$. \square

The necessity part of Theorem 5.1 now is exactly the statement below.

Proposition 5.6. Let A be a densely defined, closed and skew-symmetric operator and let (H, Γ_1, Γ_2) be a boundary triplet for A . Then $\dim \ker(1 - A^*) = \dim \ker(1 + A^*)$ holds.

Proof. We put $K = -\text{id}_H$. By Lemma 5.3 $B = \psi(K)$ satisfies $B = -B^*$ and $D(B) = \ker \Gamma_1$. We fix $\varepsilon = \pm 1$ and use Lemma 5.4 to derive

$$D(-A^*) = D(B) \oplus \ker(1 + A^*) = \ker \Gamma_1 \oplus \ker(1 + \varepsilon A^*).$$

By (BT-2) the map $\Gamma_1: \ker \Gamma_1 \oplus \ker(1 + \varepsilon A^*) \rightarrow H$ is surjective and in view of the decomposition this implies immediately that $\Gamma_1|_{\ker(1 + \varepsilon A^*)}: \ker(1 + \varepsilon A^*) \rightarrow H$ is bijective.

Since A^* is closed, $\ker(1 + \varepsilon A^*) \subseteq X$ is closed and thus a Banach space w.r.t. the norm $\|\cdot\|_X$. Since

$$\|x\|_X \leq \|x\|_X + \|-A^*x\|_X = \|x\|_X + \|\varepsilon A^*x\|_X = 2\|x\|_X$$

holds for $x \in \ker(1 + \varepsilon A^*)$, the norm $\|\cdot\|_X$ and the graph norm $\|\cdot\|_{D(-A^*)}$, see Lemma 5.5, are equivalent on $\ker(1 + \varepsilon A^*)$. Lemma 5.5 thus shows that $\Gamma_1: \ker(1 + \varepsilon A^*) \rightarrow X$ is continuous. The open mapping theorem yields that the latter is an isomorphism. \square

6. PORT-HAMILTONIAN SYSTEMS

In this section we apply the results of Section 4 to port-Hamiltonian systems. We follow Le Gorrec, Zwart, Maschke [15] but add some more details and are able to remove some technicalities due to our relation-free and throughout skew-symmetric approach above.

Let $d \geq 1$ be a fixed integer and $a < b$ be real numbers. We consider the partial differential equation

$$\frac{\partial x}{\partial t}(t, \xi) = P_0(\mathcal{H}x)(t, \xi) + P_1 \frac{\partial(\mathcal{H}x)}{\partial \xi}(t, \xi) \quad \text{for } t \geq 0 \text{ and } \xi \in (a, b)$$

where $P_0, P_1 \in \mathbb{C}^{d \times d}$ with $P_0^* = -P_0$ and $P_1^* = P_1$ invertible. We assume that $\mathcal{H}: (a, b) \rightarrow \mathbb{C}^{d \times d}$ is measurable and that there exists $0 < m \leq M$ s.th. for almost every $\xi \in (a, b)$ the matrix $\mathcal{H}(\xi)$ is self-adjoint and that we have

$$m|\zeta|^2 \leq \langle \mathcal{H}(\xi)\zeta, \zeta \rangle_{\mathbb{C}^d} \leq M|\zeta|^2$$

for all $\zeta \in \mathbb{C}^d$ where $\langle \cdot, \cdot \rangle_{\mathbb{C}^d}$ denotes the inner product of the euclidean space \mathbb{C}^d . In the sequel, we write L^2 for the space $L^2(a, b; \mathbb{C}^d)$ with the standard inner product $\langle \cdot, \cdot \rangle_{L^2}$ which is linear in the first argument. We use the letter \mathcal{H} for the operator $\mathcal{H}: L^2 \rightarrow L^2$, $(\mathcal{H}x)(\xi) = \mathcal{H}(\xi)x(\xi)$ for $\xi \in (a, b)$, which is a self-adjoint operator.

Lemma 6.1. Let \mathcal{H} be as above.

- (i) The space $(X, \langle \cdot, \cdot \rangle_X)$ with $X = L^2$ and $\langle \cdot, \cdot \rangle_X = \langle \mathcal{H}\cdot, \cdot \rangle_{L^2}$ is a Hilbert space.
- (ii) The operator $\mathcal{H}: X \rightarrow L^2$ is an isomorphism of Banach spaces.

Proof. (i) From our assumptions on \mathcal{H} it follows that $\langle \cdot, \cdot \rangle_X$ is an inner product on X and that $\|\cdot\|_{L^2}$ and $\|\cdot\|_X$ are equivalent. Therefore $(X, \|\cdot\|_X)$ is complete and thus $(X, \langle \cdot, \cdot \rangle_X)$ is Hilbert.

(ii) We first consider $\mathcal{H}: L^2 \rightarrow L^2$ and observe that this operator is continuous. By our assumptions on \mathcal{H} we have $\langle \mathcal{H}x, x \rangle_{L^2} \geq m\|x\|_{L^2}^2$ for every $x \in L^2$. This shows that \mathcal{H} is injective and we can consider $\mathcal{H}^{-1}: \text{ran } \mathcal{H} \rightarrow L^2$. For $y = \mathcal{H}x \in \text{ran } \mathcal{H}$ with $x \in L^2$ we get

$$\|\mathcal{H}^{-1}y\|_{L^2} = \|x\|_{L^2} \leq \|\mathcal{H}x\|_{L^2} = \|y\|_{L^2}$$

by the Cauchy-Schwarz-Bunjakowski inequality. Thus, \mathcal{H}^{-1} is continuous and $\text{ran } \mathcal{H} \subseteq L^2$ is closed. Let $y \in \text{ran } \mathcal{H}^\perp$, i.e., $\langle Ax, y \rangle = 0$ for all $x \in L^2$. Putting $x = y$ shows together with our initial estimate, that only $y = 0$ is possible. Thus, $\text{ran } \mathcal{H} \subseteq L^2$ is dense by [7, I.2.11]. Consequently, $\text{ran } \mathcal{H} = L^2$ and we proved that $\mathcal{H}: L^2 \rightarrow L^2$ is an isomorphism of Banach spaces. The proof of (i) showed that $\text{id}_{L^2}: X \rightarrow L^2$ is an isomorphism. A combination of both statements establishes that $\mathcal{H}: X \rightarrow L^2$ is an isomorphism. \square

We denote by H^1 the Sobolev space $H^1(a, b, \mathbb{C}^d)$ whose members are all absolutely continuous functions $x \in L^2$ —in particular x is almost everywhere differentiable—such that also $x' \in L^2$ holds. We define the operator $A_0: D(A_0) \rightarrow X$ via

$$A_0x = P_0\mathcal{H}x + P_1(\mathcal{H}x)' \quad \text{for } x \in D(A_0) = \{x \in X; \mathcal{H}x \in H^1 \text{ and } x(a) = x(b) = 0\}$$

and define $\Gamma_1, \Gamma_2: H^1 \rightarrow \mathbb{C}^d$ via

$$\Gamma_1y = \frac{1}{\sqrt{2}}P_1(y(b) - y(a)) \quad \text{and} \quad \Gamma_2y = \frac{1}{\sqrt{2}}(y(b) + y(a))$$

for $y \in H^1$. Our aim for this section is to employ the method of boundary triplets that we established in Section 4 to classify those extensions of A_0 which generate a C_0 -semigroup.

In view of the condition in Theorem 6.2 below we remind the reader at this point that for $M \in \mathbb{C}^{d \times d}$ the estimate $M \geq 0$ holds by definition if $\langle M\xi, \xi \rangle_{\mathbb{C}^d} \geq 0$ for all $\xi \in \mathbb{C}^d$.

Theorem 6.2. If $W \in \mathbb{C}^{d \times 2d}$ satisfies $\text{rk } W = d$ and $W\Sigma W^* \geq 0$ with $\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, then the operator $B_0: D(B_0) \rightarrow X$ defined by

$$B_0x = P_0\mathcal{H}x + P_1(\mathcal{H}x)' \text{ for } x \in D(B_0) = \{x \in X; \mathcal{H}x \in H^1 \text{ and } W \begin{bmatrix} \Gamma_1 \mathcal{H}x \\ \Gamma_2 \mathcal{H}x \end{bmatrix} = 0\}$$

extends the operator $A_0: D(A_0) \rightarrow X$ and generates a C_0 -semigroup of contractions. Conversely, every operator $B_0: D(B_0) \rightarrow X$, that extends $A_0: D(A_0) \rightarrow X$ and generates a C_0 -semigroup of contractions, is of the form above for some W with the properties above.

We use the following lemma in order to reduce the proof to the special case $\mathcal{H} = \text{id}_{L^2}$.

Lemma 6.3. Let \mathcal{H} and X be as above. An operator $B: D(B) \rightarrow L^2$ is maximal dissipative in L^2 if and only if the operator $B_0: D(B_0) \rightarrow X$ defined by $B_0x = B\mathcal{H}x$ for $x \in D(B_0) = \{x \in X; \mathcal{H}x \in D(B)\}$ is maximal dissipative in X .

Proof. For $x \in D(B_0)$, i.e., $\mathcal{H}x \in D(B)$, we have $\langle B_0x, x \rangle_X = \langle \mathcal{H}B\mathcal{H}x, x \rangle_{L^2} = \langle B\mathcal{H}x, \mathcal{H}x \rangle_{L^2}$. Since the isomorphism $\mathcal{H}: X \rightarrow L^2$ maps $D(B_0)$ onto $D(B)$ we get that B is densely defined if and only if B_0 is densely defined. Together with our first computation the latter shows that B_0 is dissipative in X if and only if B is dissipative in L^2 .

If C is an extension of B then $C_0: D(C_0) \rightarrow X$ defined by $C_0x = C\mathcal{H}x$ for $x \in D(C_0) = \{x \in X; \mathcal{H}x \in D(C)\}$ is an extension of B_0 . If C_0 is an extension of B_0 , then $C: D(C) \rightarrow L^2$ defined by $Cx = C_0\mathcal{H}^{-1}x$ for $x \in D(C) = \{x \in L^2; \mathcal{H}^{-1}x \in D(C_0)\}$ is an extension of B . The above assignments are inverse to each other and by the first part of the proof, the operator C is dissipative if and only if C_0 is dissipative. We thus established a one-to-one correspondence between the dissipative extensions of B and B_0 which finishes the proof. \square

By the lemma above it is enough, to prove Theorem 6.2 for the case $\mathcal{H} = \text{id}_{L^2}$. To avoid confusion, we state this special case explicitly below. First, we fix the following notation for the rest of this section. Let $A: D(A) \rightarrow L^2$ be given by

$$Ax = P_0x + P_1x' \text{ for } x \in D(A) = \{x \in H^1; x(a) = x(b) = 0\}$$

and fix $\Gamma_1, \Gamma_2: H^2 \rightarrow \mathbb{C}^d$ with

$$\Gamma_1y = \frac{1}{\sqrt{2}}P_1(y(b) - y(a)) \quad \text{and} \quad \Gamma_2y = \frac{1}{\sqrt{2}}(y(b) + y(a))$$

for $y \in H^1$. Finally we put

$$W = \{W \in \mathbb{C}^{d \times 2d}; \text{rk } W = d \text{ and } W\Sigma W^* \geq 0\} \text{ with } \Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

and have to prove the following.

Proposition 6.4. (i) If $W \in \mathcal{W}$ then the operator $B: D(B) \rightarrow L^2$ given by

$$Bx = P_0x + P_1x' \text{ for } x \in D(B) = \{x \in H^1; W \begin{bmatrix} \Gamma_1 x \\ \Gamma_2 x \end{bmatrix} = 0\}$$

generates a C_0 -semigroup of contractions.

(ii) If $B: D(B) \rightarrow X$ extends $A: D(A) \rightarrow X$ and generates a C_0 -semigroup of contractions, then there exists $W \in \mathcal{W}$ such that $B: D(B) \rightarrow L^2$ is of the form above.

We start with the following properties of the operator $A: D(A) \rightarrow L^2$ and the maps Γ_1 and Γ_2 .

Lemma 6.5. The operator $A: D(A) \rightarrow L^2$ is densely defined, closed and skew-symmetric. More precisely, $-A^*: D(-A^*) \rightarrow X$ is given by $-A^*x = P_0x + P_1x'$ for $x \in D(-A^*) = H^1$.

Proof. That $D(A) \subseteq L^2$ is dense can be seen by the same arguments used in [7, Example X.1.11].

The multiplication operators $P_k: L^2 \rightarrow L^2$, $[P_k(x)](\xi) = P_k x(\xi)$ for $\xi \in (a, b)$ and $k = 0, 1$ are continuous. The derivative $D: D(A) \rightarrow L^2$, see again [7, Example X.1.11], and thus also $P_1 \circ D$ is closed. Finally, $A = P_0 + P_1 \circ D$ is closed as the sum of a continuous and a closed operator.

We put $A_1 = P_1 \circ D$ with $D(A_1) = D(A)$. We claim that $D(A_1^*) = H^1$. Let $y \in D(A_1^*)$, put $z = P_1^{-1} A_1^* y$ and $Z(\cdot) = \int_a^\cdot z(\xi) d\xi$. For $x \in D(A_1)$ we compute

$$\int_a^b \langle x'(\xi), P_1 y(\xi) \rangle_{\mathbb{C}^d} d\xi = \langle A_1 x, y \rangle_{L^2} = \langle x, A_1^* y \rangle_{L^2} = \langle x, P_1 z \rangle_{L^2} = - \int_a^b \langle x'(\xi), P_1 Z(\xi) \rangle_{\mathbb{C}^d} d\xi$$

where the last equation follows by component wise integration by parts. We showed that $\langle x', P_1(y + Z) \rangle_{L^2} = 0$ holds for all $x \in D(A_1)$. Consequently, $P_1(y + Z) \in \{x'; x \in H^1\}^\perp = \text{span}\{1\}$ and thus there exists $c \in \mathbb{C}$ such that $P_1(y(\xi) + Z(\xi)) = c$ for all $\xi \in (a, b)$. Therefore, $y = P_1^{-1}c - Z$ is absolutely continuous and $y' = -z \in L^2$, i.e., $y \in H^1$ and in addition $A_1^* y = P_1 z = -P_1 y'$. Let now $y \in H^1$. For $x \in D(A_1)$, in particular $x(a) = x(b) = 0$, we compute

$$\langle A_1 x, y \rangle_{L^2} = \int_a^b \langle P_1 x'(\xi), y(\xi) \rangle_{\mathbb{C}^d} d\xi = - \int_a^b \langle x(\xi), P_1 y'(\xi) \rangle_{\mathbb{C}^d} d\xi = \langle x, -P_1 y' \rangle_{L^2}$$

which shows that the map $D(A_1) \rightarrow \mathbb{C}$, $x \mapsto \langle A_1 x, y \rangle_{L^2}$ is continuous, i.e., $y \in D(A_1^*)$.

Since the multiplication operator $P_0: L^2 \rightarrow L^2$ is continuous and the matrix P_0 satisfies $P_0^* = -P_0$, it follows $A^* = (P_0 + A_1)^* = P_0^* + A^*$, i.e., $D(-A^*) = H^1$ and $A^* x = -P_0 x - P_1 x'$. In particular, A is skew-symmetric. \square

Lemma 6.6. The triple $(\mathbb{C}^d, \Gamma_1, \Gamma_2)$ is a boundary triplet for A .

Proof. By Lemma 6.5 the maps $\Gamma_1, \Gamma_2: D(-A^*) \rightarrow \mathbb{C}^d$ are well-defined. We have to check the two conditions in Definition 4.1.

Let $x, y \in H^1$ be given. We compute

$$\begin{aligned} \langle A^* x, y \rangle_{L^2} + \langle x, A^* y \rangle_{L^2} &= - \int_a^b \langle P_0 x(\xi) + P_1 x'(\xi), y(\xi) \rangle_{\mathbb{C}^d} d\xi + \langle x(\xi), P_0 y(\xi) + P_1 y'(\xi) \rangle_{\mathbb{C}^d} d\xi \\ &= - \int_a^b \langle P_1 x'(\xi), y(\xi) \rangle_{\mathbb{C}^d} d\xi - \int_a^b \langle x(\xi), P_1 y'(\xi) \rangle_{\mathbb{C}^d} d\xi \\ &= \langle P_1 x(\xi), y(\xi) \rangle_{\mathbb{C}^d} \Big|_a^b + \int_a^b \langle x(\xi), P_1 y'(\xi) \rangle_{\mathbb{C}^d} d\xi - \int_a^b \langle x(\xi), P_1 y'(\xi) \rangle_{\mathbb{C}^d} d\xi \\ &= \langle P_1 x(b), y(b) \rangle_{\mathbb{C}^d} - \langle P_1 x(a), y(a) \rangle_{\mathbb{C}^d} \\ &= \frac{1}{2} [\langle P_1(x(b) - x(a)), y(b) + y(a) \rangle_{\mathbb{C}^d} + \langle x(b) + x(a), P_1(y(b) - y(a)) \rangle_{\mathbb{C}^d}] \\ &= \langle \Gamma_1 x, \Gamma_2 y \rangle_{\mathbb{C}^d} + \langle \Gamma_2 x, \Gamma_1 y \rangle_{\mathbb{C}^d} \end{aligned}$$

which establishes (BT-1). Let $y_1, y_2 \in \mathbb{C}^d$ be given. Define $x: [a, b] \rightarrow \mathbb{C}^d$ via

$$x(\xi) = \frac{1}{\sqrt{2}} (y_2 - P_1^{-1} y_1 + 2 \frac{\xi-a}{b-a} P_1^{-1} y_1)$$

for $\xi \in [a, b]$. Thus, we have $x \in H^1$, $x(a) = \frac{1}{\sqrt{2}} (y_2 - P_1^{-1} y_1)$ and $x(b) = \frac{1}{\sqrt{2}} (y_2 + P_1^{-1} y_1)$. Therefore

$$\Gamma_1 x = \frac{1}{2} P_1 (y_2 + P_1^{-1} y_1 - y_2 + P_1^{-1} y_1) = y_1 \quad \text{and} \quad \Gamma_2 x = \frac{1}{2} (y_2 + P_1^{-1} y_1 + y_2 - P_1^{-1} y_1) = y_2$$

hold. This establishes (BT-2). \square

For the rest of the section we fix the boundary triplet $(\mathbb{C}^d, \Gamma_1, \Gamma_2)$. Below we apply the results of Section 4 to this triplet. In particular, the abstract space H appearing in the definition of \mathcal{K} in Section 4 is from now always the Hilbert space \mathbb{C}^d .

Lemma 6.7. The map $\theta: \mathcal{W} \rightarrow \mathcal{K}$, $\theta(W) = -(W_1 + W_2)^{-1} (W_1 - W_2)$, for $W = [W_1 \ W_2]$ with $W_1, W_2 \in \mathbb{C}^{d \times d}$, is well-defined, surjective and

$$D(\psi \circ \theta(W)) = \{x \in H^1; W \begin{bmatrix} \Gamma_1 x \\ \Gamma_2 x \end{bmatrix} = 0\}$$

holds for every $W \in \mathcal{W}$.

Proof. We remember that the estimate $M_1 \geq M_2$ for $M_1, M_2 \in \mathbb{C}^{d \times d}$ means by definition that $M_1 - M_2 \geq 0$ holds; $M \geq 0$ for a matrix M means $\langle M\xi, \xi \rangle_{\mathbb{C}^d} \geq 0$ for all $\xi \in \mathbb{C}^d$.

Let $W = [W_1 \ W_2] \in \mathcal{W}$ with $W_1, W_2 \in \mathbb{C}^{d \times d}$ be given. We compute $W\Sigma W^* = W_1 W_2^* + W_2 W_1^* \geq 0$, that is $W_2 W_1^* \geq -W_1 W_2^*$ and $W_1 W_2^* \geq -W_2 W_1^*$. Therefore,

$$\begin{aligned} (W_1 + W_2)(W_1 + W_2)^* &= W_1 W_2^* + W_1 W_2^* + W_2 W_1^* + W_2 W_2^* \\ &\geq W_1 W_2^* - W_2 W_1^* - W_1 W_2^* + W_2 W_2^* \\ &= (W_1 - W_2)(W_1 - W_2)^* \\ &\geq 0. \end{aligned}$$

Let $\zeta \in \mathbb{C}^d$ and assume $W_1^* \zeta + W_2^* \zeta = 0$. By the above $\langle (W_1 - W_2)^* \zeta, (W_1 - W_2)^* \zeta \rangle_{\mathbb{C}^d} = 0$, i.e., $W_1^* \zeta - W_2^* \zeta = 0$. Both equations together yield $W_1^* \zeta = W_2^* \zeta = 0$, i.e., $W\zeta = [W_1 \ W_2]\zeta = 0$. Since $\text{rk } W = d$ it follows $\zeta = 0$ and we proved that $W_1 + W_2$ and thus also $(W_1 + W_2)^*$ is invertible. Using the estimate above again, we get

$$\theta(W)\theta(W)^* = (W_1 + W_2)^{-1}(W_1 - W_2)(W_1 - W_2)^*((W_1 + W_2)^*)^{-1} \leq \text{id}_{\mathbb{C}^d}$$

which shows $\|\theta(W)\zeta\|_{\mathbb{C}^d}^2 = |\langle \theta(W)\theta(W)^* \zeta, \zeta \rangle_{\mathbb{C}^d}| \leq 1$ for every $\zeta \in \mathbb{C}^d$, i.e., $\theta(W) \in \mathcal{K}$. Now we compute

$$W = \frac{1}{2}[W_1 + W_2 + W_1 - W_2 \quad W_1 + W_2 - W_1 + W_2] = S[\theta(W) - \text{id}_{\mathbb{C}^d} \quad -(\theta(W) + \text{id}_{\mathbb{C}^d})]$$

with $S = -\frac{1}{2}(W_1 + W_2)$. Since S is invertible, we have

$$D(\psi \circ \theta(W)) = \{x \in H^1; [\theta(W) - \text{id}_{\mathbb{C}^d} \quad -(\theta(W) + \text{id}_{\mathbb{C}^d})] \begin{bmatrix} \Gamma_{1x} \\ \Gamma_{2x} \end{bmatrix} = 0\} = \{x \in H^1; W \begin{bmatrix} \Gamma_{1x} \\ \Gamma_{2x} \end{bmatrix} = 0\}.$$

Let $K \in \mathcal{K}$ be given. We put $W = [K - \text{id}_{\mathbb{C}^d} \quad -(K + \text{id}_{\mathbb{C}^d})]$ and compute

$$\theta(W) = -(K - \text{id}_{\mathbb{C}^d} - (K + \text{id}_{\mathbb{C}^d}))^{-1}(K - \text{id}_{\mathbb{C}^d} + (K + \text{id}_{\mathbb{C}^d})) = K.$$

Moreover we have $W\Sigma W^* = 2(\text{id}_{\mathbb{C}^d} - KK^*) \geq 0$ since $K \in \mathcal{K}$ implies $KK^* \leq \text{id}_{\mathbb{C}^d}$. Let $\zeta \in \mathbb{C}^d$ be given with

$$W^* \zeta = [(K - \text{id}_{\mathbb{C}^d})\zeta \quad -(K + \text{id}_{\mathbb{C}^d})\zeta] = 0.$$

Thus, $(K - \text{id}_{\mathbb{C}^d})\zeta = 0$ and $(K + \text{id}_{\mathbb{C}^d})\zeta = 0$. Consequently, $\zeta = 0$ and we obtain that $\text{rk } W = \text{rk } W^* = d$. This establishes the surjectivity of θ . \square

Now we apply the results of Section 4. We remember that we already fixed a boundary triplet and that all symbols introduced in Section 4 have to be understood with respect to this particular triplet.

Proof. (of Proposition 6.4) Let $W \in \mathcal{W}$ be given. By Lemma 6.7, we have $\theta(W) \in \mathcal{K}$ with

$$D(\psi \circ \theta(W)) = \{x \in H^1; W \begin{bmatrix} \Gamma_{1x} \\ \Gamma_{2x} \end{bmatrix} = 0\} = D(B)$$

and $\psi \circ \theta(W) \in \mathcal{D}$. By Proposition 2.8 therefore $\psi \circ \theta(W) \subseteq -A^*$ holds. Lemma 6.5, together with the equality of domains that we established already, yields

$$\psi \circ \theta(W)x = P_0 x + P_1 x' = Bx$$

for $x \in D(\psi \circ \theta(W)) = D(B)$.

For the second part let $B: D(B) \rightarrow L^2$ be an extension of A that generates a C_0 -semigroup of contractions. By the Lumer-Phillips theorem, see Section 3, B is a maximal dissipative extension of A , i.e., $B \in \mathcal{D}$. By Lemma 6.7 there exists $W \in \mathcal{W}$ such that $\theta(W) = \phi(B) \in \mathcal{K}$ holds, from whence the equality

$$B = \psi \circ \phi(B) = \psi \circ \theta(W)$$

follows. The first part of the proof shows that B is of the desired form. \square

7. NOTES AND REFERENCES

Below we give the references to those papers and books in which the results presented in the preceding sections have been originally proved. In addition, we comment these results and the methods we used for their proofs. We give detailed references and point out alternative approaches.

Section 2.

Our notion of a dissipative operator in Definition 2.1(i) is widely used in the literature on C_0 -semigroups, cf. Engel, Nagel [13] and Pazy [21]. In the symmetric situation, and in particular in the book [14] by Gorbachuk, Gorbachuk, which is often used as a reference for the theory of boundary triplets, A is by definition dissipative if $\operatorname{Im}\langle Ax, x \rangle \geq 0$ holds for all $x \in D(A)$. This corresponds to passing from the operator A to iA . We like to refer also to the book [25] by Sz.-Nagy, Foias, Bercovici, Kérchy, in particular to [25, IV.4].

Our terminology of maximal dissipativity in Definition 2.1(ii) is defined via extensions. In [21, Definition 3.3.1] the notion of m -dissipative operators is used. An operator A is *m -dissipative* if A is dissipative and $\lambda - A$ is surjective for some or, equivalently, all $\lambda > 0$. The relation between these two, a priori different, notions depends on the underlying situation. In the previous sections the operator A under consideration was always assumed to be densely defined. Dissipative, maximal dissipative and m -dissipative operators can however be studied without this assumption. Indeed, even if X is only a Banach space, one can define the three notions: An operator A is then by definition dissipative if the condition in Lemma 2.2(iv) holds, cf. [13, Definition II.3.13]. If now A is densely defined, then A is maximal dissipative if and only if A is m -dissipative. For Hilbert spaces this follows from the material presented in Section 3 and our comments below. For Banach spaces we refer to [13, Definition II.3.13 and Proposition II.3.23] together with [21, Section 1.4 and p. 81]. If, on the other hand, X is a Hilbert space then every m -dissipative operator is automatically densely defined, in fact this is even true for reflexive Banach spaces [13, Corollary II.3.20]. In this situation we thus derive that every m -dissipative operator is maximal dissipative. The converse is finally not true even if X is Hilbert: The results [22, footnote on p. 201] provide an example of a maximal dissipative operator which is not densely defined by [22, Lemma 1.1.3] and consequently cannot be m -dissipative.

The decomposition in Lemma 2.5 can alternatively be proved by adapting the proof of [7, Lemma X.2.13] where symmetric operators are considered. Our proof is an adaption of [24, Proposition 3.7].

Section 3.

The version of the Lumer-Phillips theorem in Section 3 is adjusted to our definition of maximal dissipativity. A popular alternative is the following [21, Theorem 4.3]: If A is densely defined then the following are equivalent.

- (i) A is dissipative and $\lambda - A$ is surjective for some or, equivalently, for all $\lambda > 0$,
- (ii) A generates a C_0 -semigroup of contractions.

In [13, Section II.3.b] more sophisticated versions can be found. The reader familiar with the version stated above will find it easy to extract the equivalence of

- (iii) A is maximal dissipative in the sense of Definition 2.1(i),
- and (ii) from the proof given in Section 3.

Section 4 and Section 5.

For precise historical information on the symmetric counterparts of the statements explained in Section 2–5 we refer to Gorbachuk, Gorbachuk [14, p. 320].

Let X be a Hilbert space and $A: D(A) \rightarrow X$ be a closed, densely defined and skew-symmetric operator. Define $L: \operatorname{ran}(1 - A) \rightarrow X$ via $L(x - Ax) = x + Ax$. Then by Phillips [22, p. 199–201] there is a one-to-one correspondence

$$\left\{ \begin{array}{l} B: D(B) \rightarrow X \\ A \subseteq B \text{ \& } B \text{ dissipative} \end{array} \right\} \xleftrightarrow[\Psi]{\Phi} \left\{ \begin{array}{l} K: D(K) \rightarrow X \\ L \subseteq K \text{ \& } K \text{ contractive} \end{array} \right\} \quad (5)$$

where the maps ϕ and ψ are given by $\phi(B): \text{ran}(1 - B) \rightarrow X$, $\phi(B)(x - Bx) = x + Bx$ and $\psi(K): \text{ran}(1 + K) \rightarrow X$, $\psi(K)(Kx + x) = Kx - x$. The maximal dissipative extensions correspond exactly to those contractions which are defined on the whole space. The method of boundary triplets can be regarded as a refinement of this classical result: In (5) the contractions which are used to parametrize the dissipative extensions are defined on the initial Hilbert space X , which is typically an infinite dimensional Hilbert space. But the above suggests that for the parametrization a smaller space may be enough. Since $\phi(B)|_{\text{ran}(1-A)} = L$ holds for every extension B of A it is reasonable to give the space $X/\text{ran}(1 - A) = \ker(1 - A^*) =: H$ a try as domain of the contractions. Killing those part of the domains on which there is no choice for the values of B means to consider $D(-A^*)/D(A) = \ker(1 - A^*) \oplus \ker(1 + A^*)$, cf. Lemma 2.5. This discrepancy gives rise to the characterization that the re-parametrization works if $\ker(1 - A^*) \cong \ker(1 + A^*)$ holds. The precise details are contained in Section 5. Note that in the book [14] by Gorbachuk, Gorbachuk only the sufficiency part of Theorem 5.1 is given; the necessity it taken from Schmüdgen [24, Proposition 14.5].

Section 6.

We refer to the book [17] by Jacob, Zwart for a detailed introduction to port-Hamiltonian systems which includes also an introduction to semigroup theory and many examples and applications from control and systems theory. The exposition in [17] avoids the notion of boundary triplets and uses a direct Lumer-Phillips argument. The technical ingredients for this are however the same as in the approach given in Section 6. The latter follows mainly from Le Gorrec, Zwart, Maschke [15], only some details are different: Firstly, in the latter paper the differential operator is of order $N \geq 1$ whereas we restricted ourselves to the case $N = 1$ to make the computations somewhat easier. Secondly, in the [15] the matrices P_i are real and in our case they are complex. For further literature, in particular on applications of port-Hamiltonian systems in control theory, we refer to the references given in Section 1. In particular, we refer to [15, 17] for physical interpretations which show that the special form of boundary conditions that are used in Theorem 6.2 is reasonable for applications.

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